# Asymptotic formulas for eigenvalues and eigenfunctions of a new boundary-value-transmission problem

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**Abstract**

In this paper we are concerned with a new class of BVP' s consisting of eigendependent boundary conditions and two supplementary transmission conditions at one interior point. By modifying some techniques of classical Sturm-Liouville theory and suggesting own approaches we find asymptotic formulas for the eigenvalues and eigenfunction.

*Keywords:* Sturm-Liouville problems, eigenvalue, eigenfunction, asymptotics of eigenvalues and eigenfunction.

## 1. Introduction

Boundary value problems arise directly as mathematical models of motion according to Newton's law, but more often as a result of using the method of separation of variables to solve the classical partial differential equations of physics, such as Laplace's equation, the heat equation, and the wave equation. Many topics in mathematical physics require investigations of eigenvalues and eigenfunctions of boundary value problems. These investigations are of utmost importance for theoretical and applied problems in mechanics, the theory of vibrations and stability, hydrodynamics, elasticity, acoustics, electrodynamics, quantum mechanics, theory of systems and their optimization, theory of random processes, and many other branches of natural science. Such problems are formulated in many different ways.

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In this study we shall investigate a new class of Sturm-Liouville type problem which consist of a Sturm-Liouville equation contained

$$\mathcal{L}(y) := -p(x)y''(x) + q(x)y(x) = \mu^2 y(x) \tag{1}$$

to hold in finite interval $(a, b)$ except at one inner point $c \in (a, b)$, where discontinuity in $u$ and $u'$ are prescribed by two-point eigenparameter- dependent boundary conditions

$$V_1(y) := \alpha_{10}y(a) - \alpha_{11}y'(a) - \mu^2(\alpha'_{10}y(a) - \alpha'_{11}y'(a)) = 0, \tag{2}$$

$$V_2(y) := \alpha_{20}y(b) - \alpha_{21}y'(b) + \mu^2(\alpha'_{20}y(b) - \alpha'_{21}y'(b)) = 0, \tag{3}$$

together with the transmission conditions

$$V_j(y) := \beta^-_{j1}y'(c-) + \beta^-_{j0}y(c-) + \beta^+_{j1}y'(c+) + \beta^+_{j0}y(c+) = 0, \quad j = 1, 2 \tag{4}$$

where $p(x) = p^- > 0$ for $x \in [a, c)$, $p(x) = p^+ > 0$ for $x \in (c, b]$, the potential $q(x)$ is real-valued function which continuous in each of the intervals $[a, c)$ and $(c, b]$, and has a finite limits $q(c \mp 0)$, $\lambda$ is a physical complex parameter, $\alpha_{ij}$, $\beta^{\pm}_{ij}$, $\alpha'_{ij}$ $(i = 1, 2$ and $j = 0, 1)$ are real numbers. We describe some analytical solutions of the problem and find asymptotic formulas of eigenvalues and eigenfunctions. These boundary conditions are of great importance for theoretical and applied studies and have a definite mechanical or physical meaning (for instance, of free ends). Also the problems with transmission conditions arise in mechanics, such as thermal conduction problems for a thin laminated plate, which studied in [9]. This class of problems essentially differs from the classical case, and its investigation requires a specific approach based on the method of separation of variables. Moreover the eigenvalue parameter appear in one of the boundary conditions and two new conditions added to boundary conditions called transmission conditions.

## 2. The fundamental solutions and characteristic Function

Let $B_0 = \begin{bmatrix} \alpha_{11} & \alpha_{10} \\ \alpha'_{11} & \alpha'_{10} \end{bmatrix}$, $B_1 = \begin{bmatrix} \alpha_{21} & \alpha_{20} \\ \alpha'_{21} & \alpha'_{20} \end{bmatrix}$ and T$=\begin{bmatrix} \beta^-_{10} & \beta^-_{11} & \beta^+_{10} & \beta^+_{11} \\ \beta^-_{20} & \beta^-_{21} & \beta^+_{20} & \beta^+_{21} \end{bmatrix}$. Denote the determinant of the matrix $B_0$ by $\theta_1$, the determinant of the matrix $B_1$ by $\theta_2$ and the determinant of the k-th and j-th columns of the matrix T by $\Delta_{kj}$. Note that throughout this study we shall assume that $\theta_1 > 0$, $\theta_2 > 0$, $\Delta_{12} > 0$ and $\Delta_{34} > 0$. With a view to constructing the

characteristic function we define two solution $\psi(x,\mu)$ and $\varphi(x,\mu)$ as follows. Denote the solutions of the equation 1 satisfying the initial conditions

$$y(a)=\alpha_{11}-\mu^2\alpha'_{11},\ y'(a)=\alpha_{10}-\mu^2\alpha'_{10} \tag{5}$$

and

$$y(b)=\alpha_{21}+\mu^2\alpha'_{21},\ y'(b)=\alpha_{20}+\mu^2\alpha'_{20} \tag{6}$$

by $u=\varphi^-(x,\mu)$ and $u=\psi^+(x,\mu)$, respectively. It is known that the initial-value problems has an unique solutions $u=\varphi^-(x,\mu)$ and $u=\psi^+(x,\mu)$, which is an entire function of $\mu\in\mathbb{C}$ for each fixed $x\in[a,c)$ and $x\in(c,b]$ respectively. (see, for example, [10]). Using this solutions we can prove that the equation (1) on $[a,c)$ and $x\in(c,b]$ has solutions $u=\varphi^+(x,\mu)$ and $u=\psi^-(x,\mu)$, satisfying the initial conditions

$$y(c)=\frac{1}{\Delta_{12}}(\Delta_{23}\varphi^-(c,\mu)+\Delta_{24}\frac{\partial\varphi^-(c,\mu)}{\partial x}) \tag{7}$$

$$y'(c)=\frac{-1}{\Delta_{12}}(\Delta_{13}\varphi^-(c,\mu)+\Delta_{14}\frac{\partial\varphi^-(c,\mu)}{\partial x}). \tag{8}$$

and

$$y(c)=\frac{-1}{\Delta_{34}}(\Delta_{14}\psi^+(c,\mu)+\Delta_{24}\frac{\partial\psi^+(c,\mu)}{\partial x}), \tag{9}$$

$$y'(c)=\frac{1}{\Delta_{34}}(\Delta_{13}\psi^+(c,\mu)+\Delta_{23}\frac{\partial\psi^+(c,\mu)}{\partial x}). \tag{10}$$

respectively. Consequently, the solution $u=\varphi(x,\mu)$ defined by

$$\varphi(x,\mu)=\begin{cases}\varphi^-(x,\mu), & x\in[a,c)\\ \varphi^+(x,\mu), & x\in(c,b]\end{cases} \tag{11}$$

satisfies the equation (1) on $[a,c)\cup(c,b]$, the first boundary condition (2) and both transmission conditions (4).
Similarly, the solution $\psi(x,\mu)$ defined by

$$\psi(x,\mu)=\begin{cases}\psi^-(x,\mu), & x\in[a,c)\\ \psi^+(x,\mu), & x\in(c,b]\end{cases} \tag{12}$$

satisfies the equation (1) on whole $[a,c)\cup(c,b]$, the second boundary condition of (3) and both transmission condition (4). Again, similarly to [8] it can be proven that, there are infinitely many eigenvalues $\mu_n$, $n=1,2,...$ of the BVTP (1) – (4) which are coincide with the zeros of characteristic function $w(\mu)$.

## 3. Some asymptotic approximation formulas for fundamental solutions

Below, for shorting we shall use also notations; $\varphi^{\pm}(x,\mu):=\varphi^{\pm}_{\mu}(x), \psi^{\pm}(x,\mu):=\psi^{\pm}_{\mu}(x)$. By applying the method of variation of parameters we can prove that the next integral and integro-differential equations are hold for $k=0$ and $k=1$.

$$\frac{d^k}{dx^k}\varphi^{-}_{\mu}(x)=\sqrt{p^-}\frac{(\alpha_{10}-\mu^2\alpha'_{10})}{\mu}\frac{d^k}{dx^k}\sin\left[\frac{\mu\,(x-a)}{\sqrt{p^-}}\right]$$

$$+(\alpha_{11}-\mu^2\alpha'_{11})\frac{d^k}{dx^k}\cos\left[\frac{\mu\,(x-a)}{\sqrt{p^-}}\right]+\frac{1}{\sqrt{p^-}\mu}\int\limits_a^x\frac{d^k}{dx^k}\sin\left[\frac{\mu\,(x-z)}{\sqrt{p^-}}\right]q(z)\varphi^{-}_{\mu}(z)dz$$

$$\begin{aligned}\frac{d^k}{dx^k}\psi^{-}_{\mu}(x) &= -\frac{1}{\Delta_{34}}(\Delta_{14}\psi^{+}(c,\mu)+\Delta_{24}\frac{\partial\psi^{+}(c,\mu)}{\partial x}))\frac{d^k}{dx^k}\cos\left[\frac{\mu(x-c)}{\sqrt{p^-}}\right]\\ &+\frac{\sqrt{p^-}}{\mu\Delta_{34}}(\Delta_{13}\psi^{+}(c,\mu)+\Delta_{23}\frac{\partial\psi^{+}(c,\mu)}{\partial x}))\frac{d^k}{dx^k}\sin\left[\frac{\mu(x-c)}{\sqrt{p^-}}\right]\\ &+\frac{1}{\sqrt{p^-}\mu}\int\limits_x^{c-}\frac{d^k}{dx^k}\sin\left[\mu\,(x-z)\right]q(z)\psi^{-}_{\mu}(z)dz \end{aligned}\qquad (13)$$

for $x\in[a,c)$ and

$$\begin{aligned}\frac{d^k}{dx^k}\varphi^{+}_{\mu}(x) &= \frac{1}{\Delta_{12}}(\Delta_{23}\varphi^{-}(c,\mu)+\Delta_{24}\frac{\partial\varphi^{-}(c,\mu)}{\partial x}))\frac{d^k}{dx^k}\cos\left[\frac{\mu(x-c)}{\sqrt{p^+}}\right]\\ &-\frac{\sqrt{p^+}}{\mu\Delta_{12}}(\Delta_{13}\varphi^{-}(c,\mu)+\Delta_{14}\frac{\partial\varphi^{-}(c,\mu)}{\partial x}))\frac{d^k}{dx^k}\sin\left[\frac{\mu(x-c)}{\sqrt{p^+}}\right]\\ &+\frac{1}{\sqrt{p^+}\mu}\int\limits_{c+}^{x}\frac{d^k}{dx^k}\sin\left[\frac{\mu\,(x-z)}{\sqrt{p^+}}\right]q(z)\varphi^{+}_{\mu}(z)dz \end{aligned}\qquad (14)$$

$$\frac{d^k}{dx^k}\psi^{+}_{\mu}(x)=\frac{\sqrt{p^+}}{\mu}(\alpha_{20}+\mu^2\alpha'_{20})\frac{d^k}{dx^k}\sin\left[\frac{\mu\,(x-b)}{\sqrt{p^+}}\right]$$

$$+(\alpha_{21}+\mu^2\alpha'_{21})\frac{d^k}{dx^k}\cos\left[\frac{\mu\,(x-b)}{\sqrt{p^+}}\right]+\frac{1}{\sqrt{p^+}\mu}\int\limits_x^b\frac{d^k}{dx^k}\sin\left[\frac{\mu\,(x-z)}{\sqrt{p^+}}\right]q(z)\psi^{+}_{\mu}(z)dz$$

for $x\in(c,b]$. Now we are ready to prove the following theorems.

**Theorem 1.** *Let , $Im\mu = t$. Then if $\alpha'_{11} \neq 0$ the estimates*

$$\begin{aligned}
\frac{d^k}{dx^k}\varphi_\mu^-(x) &= -\alpha'_{11}\mu^2\frac{d^k}{dx^k}\cos\left[\frac{\mu\,(x-a)}{\sqrt{p^-}}\right] + O\left(|\mu|^{k+1}\,e^{\frac{|t|(x-a)}{\sqrt{p^-}}}\right) &(15)\\
\frac{d^k}{dx^k}\varphi_\mu^+(x) &= \frac{\Delta_{24}}{\Delta_{12}}\frac{\alpha'_{11}}{\sqrt{p^-}}\mu^3\sin\left[\frac{\mu\,(c-a)}{\sqrt{p^-}}\right]\frac{d^k}{dx^k}\cos\left[\frac{\mu\,(x-c)}{\sqrt{p^+}}\right]\\
&\quad + O\left(|\mu|^{k+2}e^{|t|(\frac{(x-c)}{\sqrt{p^+}}+\frac{(c-a)}{\sqrt{p^-}})}\right) &(16)
\end{aligned}$$

*are valid as $|\mu| \to \infty$, while if $\alpha'_{11} = 0$ the estimates*

$$\begin{aligned}
\frac{d^k}{dx^k}\varphi_\mu^-(x) &= -\alpha'_{10}\sqrt{p^-}\mu\frac{d^k}{dx^k}\sin\left[\frac{\mu(x-a)}{\sqrt{p^-}}\right] + O\left(|\mu|^{k}\,e^{\frac{|t|(x-a)}{\sqrt{p^-}}}\right) &(17)\\
\frac{d^k}{dx^k}\varphi_\mu^+(x) &= -\frac{\Delta_{24}}{\Delta_{12}}\alpha'_{10}\mu^2\cos\left[\frac{\mu\,(c-a)}{\sqrt{p^-}}\right]\frac{d^k}{dx^k}\cos\left[\frac{\mu(x-c)}{\sqrt{p^+}}\right]\\
&\quad + O\left(|\mu|^{k+1}\,e^{|t|(\frac{(x-c)}{\sqrt{p^+}}+\frac{(c-a)}{\sqrt{p^-}})}\right) &(18)
\end{aligned}$$

*are valid as $|\mu| \to \infty$ $(k = 0, 1)$. All asymtotic estimates are uniform with respect to $x$.*

Proof. The asymptotic formulas for (15) in (16) follows immediately from the Titchmarsh's Lemma on the asymptotic behavior of $\varphi_\mu^-(x)$ ([10], Lemma 1.7). But the corresponding formulas for $\varphi_\mu^+(x)$ need individual consideration. Let $\alpha_{11} \neq 0$. Put $\varphi_\mu^+(x) = e^{-|t|(x-a)}Y(x,\mu)$ it follows from (14) that

$$\begin{aligned}
Y(x,\mu) &= \frac{1}{\Delta_{12}}\alpha_{11}e^{-|t|(x-a)}[\Delta_{23}\cos s(c-a)\cos s(x-c) - \Delta_{24}s\sin s(c-a)\cos s(x-c)\\
&\quad -\frac{\Delta_{13}}{s}\cos s(c-a)\sin s(x-c) + \frac{\Delta_{14}}{s}\sin s(c-a)\sin s(x-c)]\\
&\quad +\frac{1}{s}\int_{c+}^{x}\sin\left[s(x-z)\right]q(z)e^{-|t|(x-z)}Y(z,\mu)dz + O(1) &(19)
\end{aligned}$$

Denoting $Y(\mu) = \max_{x\in(c,b]}|Y(x,\mu)|$ and $\widetilde{q} = \int\limits_{c+}^{b}|q(z)|dz$ from the last equation we have

$$Y(\mu) \leq |\frac{\Delta_{23}}{\Delta_{12}}\alpha_{11}| + |\frac{\Delta_{24}}{\Delta_{12}}\alpha_{11}| + |\frac{\Delta_{13}}{\Delta_{12}}\alpha_{11}| + |\frac{\Delta_{14}}{\Delta_{12}}\alpha_{11}| + \frac{\widetilde{q}}{a_2|s|}Y(\mu) + M$$

for some $M > 0$. Consequently $Y(\mu) = O(1)$ as $|\mu| \to \infty$, so

$$\varphi_\mu^+(x) = O(e^{|t|(x-a)}). \tag{20}$$

Consequently, the estimate (16) for the case $k = 0$ are obtained by substituting (20) in the integrals on the right-hand side of (13) . The case $k = 1$ of the (16) follows at once on differentiating (14) and making the same procedure as in the case $k = 0$. The proof of (17) in (18) is similar.

Similarly we can easily obtain the following Theorem for $\psi_\mu^\pm(x)$.

**Theorem 2.** *Let $Im\mu = t$. Then if $\alpha'_{21} \neq 0$*

$$\begin{aligned}
\frac{d^k}{dx^k}\psi_\mu^+(x) &= \alpha'_{21}\mu^2\frac{d^k}{dx^k}\cos\left[\frac{\mu\,(b-x)}{\sqrt{p^+}}\right] + O\left(|\mu|^{k+1}\, e^{\frac{|t|(b-x)}{\sqrt{p^+}}}\right) && (21)\\
\frac{d^k}{dx^k}\psi_\mu^-(x) &= -\frac{\Delta_{24}}{\Delta_{34}}\frac{\alpha'_{21}}{\sqrt{p^+}}\mu^3\sin\left[\frac{\mu\,(b-c)}{\sqrt{p^+}}\right]\frac{d^k}{dx^k}\cos\left[\frac{\mu\,(x-c)}{\sqrt{p^-}}\right] \\
&\quad + O\left(|\mu|^{k+2}e^{|t|(\frac{(b-c)}{\sqrt{p^+}}+\frac{(c-x)}{\sqrt{p^-}})}\right) && (22)
\end{aligned}$$

*as $|\mu| \to \infty$, while if $\alpha'_{21} = 0$*

$$\begin{aligned}
\frac{d^k}{dx^k}\psi_\mu^+(x) &= -a'_{20}\sqrt{p^+}\mu\frac{d^k}{dx^k}\sin\left[\frac{\mu(b-x)}{\sqrt{p^+}}\right] + O\left(|\mu|^{k}\, e^{|t|\frac{(b-x)}{\sqrt{p^+}}}\right) && (23)\\
\frac{d^k}{dx^k}\psi_\mu^-(x) &= -\frac{\Delta_{24}}{\Delta_{34}}\alpha'_{20}\mu^2\cos\left[\frac{\mu\,(b-c)}{\sqrt{p^+}}\right]\frac{d^k}{dx^k}\cos\left[\frac{\mu(x-c)}{\sqrt{p^-}}\right] \\
&\quad + O\left(|\mu|^{k+1}\, e^{|t|(\frac{(b-c)}{\sqrt{p^+}}+\frac{(c-x)}{\sqrt{p^-}})}\right) && (24)
\end{aligned}$$

*as $|\mu| \to \infty$ $(k = 0, 1)$. Each of this asymptotic equalities hold uniformly for $x$.*

## 4. Asymptotic behaviour of eigenvalues and corresponding eigenfunctions

It is well-known from ordinary differential equation theory that the Wronskians $W[\varphi_\mu^-(x), \psi_\mu^-(x)]$ and $W[\varphi_\mu^+(x), \psi_\mu^+(x)]$ are independent of variable $x$.

By using (7)-(8) and (9)-(10) we have

$$\begin{aligned} w^+(\mu) &= \varphi^+(c,\mu)\frac{\partial\psi^+(c,\mu)}{\partial x} - \frac{\partial\varphi^+(c,\mu)}{\partial x}\psi^+(c,\mu) \\ &= \frac{\Delta_{34}}{\Delta_{12}}(\varphi^-(c,\mu)\frac{\partial\psi^-(c,\mu)}{\partial x} - \frac{\partial\varphi^-(c,\mu)}{\partial x}\psi^-(c,\mu)) \\ &= \frac{\Delta_{34}}{\Delta_{12}}w^-(\mu). \end{aligned}$$

Denote $w(\mu) := \Delta_{34}w^-(\mu) = \Delta_{12}\ w^+(\mu)$.

**Theorem 3.** *The eigenvalues of the problem* (1)-(4) *are consist of the zeros of the function* $w(\mu)$*.*

Proof. Let $w(\mu_0) = 0$. Then $W[\varphi^-_{\mu_0}, \psi^-_{\mu_0}]_x = 0$. Thus, the functions $\varphi_{1\mu_0}(x)$ and $\psi^-_{\mu_0}(x)$ are linearly depended, i.e.,

$$\psi_{1\mu_0}(x) = k\varphi_{\mu_0}(x), x \in [-1, 0] \tag{25}$$

for some $k \neq 0$. In view of this equality $\psi_{\mu_0}(x)$ satisfies also the first boundary condition (2). Recall that the solution $\psi_{\mu_0}(x)$ also satisfies the second boundary condition (3) and both transmission conditions (4) Consequently $\psi_{\mu_0}(x)$ is an eigenfunction of the problem(1)- (4) corresponding to the eigenvalue $\mu_0$. Hence each zero of $w(\mu)$ is an eigenvalue.

Now let $y_0(x)$ be any eigenfunction corresponding to eigenvalue $\mu_0$. Suppose, that $w(\mu_0) \neq 0$. Then the couples of the functions $\varphi^-, \psi^-$ and $\varphi^+, \psi^+$ would be linearly independent on $[a, c)$ and $(c, b]$ respectively. Therefore, the solution $y_0(x)$ may be represented in the form

$$y_0(x) = \begin{cases} k_{11}\varphi^-_{\mu_0}(x) + k_{12}\psi^-_{\mu_0}(x) \text{ for } x \in [a, c) \\ k_{21}\varphi^+_{\mu_0}(x) + k_{22}\psi^+_{\mu_0}(x) \text{ for } x \in c, b] \end{cases} \tag{26}$$

where at least one of the coefficients $k_{11}, k_{12}, k_{21}$ and $k_{22}$ is not zero. Considering the equations

$$\tau_i(y_0(x)) = 0, \quad i = 1, 2, 3, 4 \tag{27}$$

as the homogenous system of linear equations of the variables $k_{11}, k_{12}, k_{21}, k_{22}$ and taking into account the conditions $(2)-(4)$ we obtain homogenous linear simultaneous equation of the variables $k_{ij}(i, j = 1, 2)$ the determinant of which is equal to $\frac{1}{\Delta_{12}\Delta_{34}}\omega^3(\mu)$ and therefore does not vanish by assumption. Consequently this linear simultaneous equation has the only trivial solution $k_{ij} = 0(i, j = 1, 2)$. We thus arrive at a contradiction, which completes the proof.

Now by modifying the standard method we prove that all eigenvalues of the problem $(1)-(4)$ are real.

**Theorem 4.** *The eigenvalues of the boundary value transmission problem* $(1)-(4)$ *are real.*

PROOF.

Since the Wronskians of $\varphi_\mu^+(x)$ and $\psi_\mu^+(x)$ are independent of $x$, in particular, by putting $x=a$ we have

$$\begin{aligned}\omega(\mu) &= \Delta_{34}\{\varphi^-(a,\mu)\frac{\partial\psi^-(a,\mu)}{\partial x}-\frac{\partial\varphi^-(a,\mu)}{\partial x}\psi^-(a,\mu)\} \\ &= \Delta_{34}\{(\alpha_{11}-\mu^2\alpha'_{11})\frac{\partial\psi^-(a,\mu)}{\partial x}+(\alpha_{10}-\mu^2\alpha'_{10})\psi^-(a,\mu)\}. \quad (28)\end{aligned}$$

Let $Im\mu=t$. By substituting (21) and (24) in (28) we obtain easily the following asymptotic representations
**(i)** If $\alpha'_{21}\neq 0$ and $\alpha'_{11}\neq 0$, then

$$w(\mu)=-\frac{\Delta_{24}\alpha'_{11}\alpha'_{21}}{\sqrt{p^-}\sqrt{p^+}}\mu^6\sin\left[\frac{\mu\,(c-a)}{\sqrt{p^-}}\right]\sin\left[\frac{\mu\,(b-c)}{\sqrt{p^+}}\right]+O\left(|\mu|^5\,e^{|t|(\frac{(b-c)}{\sqrt{p^+}}+\frac{(c-a)}{\sqrt{p^-}})}\right) \quad (29)$$

**(ii)** If $\alpha'_{21}\neq 0$ and $\alpha'_{11}=0$, then

$$w(\mu)=-\frac{\Delta_{24}\alpha'_{10}\alpha'_{21}}{\sqrt{p^+}}\mu^5\cos\left[\frac{\mu\,(c-a)}{\sqrt{p^-}}\right]\sin\left[\frac{\mu\,(b-c)}{\sqrt{p^+}}\right]+O\left(|\mu|^4\,e^{|t|(\frac{(b-c)}{\sqrt{p^+}}+\frac{(c-a)}{\sqrt{p^-}})}\right) \quad (30)$$

**(iii)** If $\alpha'_{21}=0$ and $\alpha'_{11}\neq 0$, then

$$w(\mu)=-\frac{\Delta_{24}\alpha'_{11}\alpha'_{20}}{\sqrt{p^-}}\mu^5\sin\left[\frac{\mu\,(c-a)}{\sqrt{p^-}}\right]\cos\left[\frac{\mu\,(b-c)}{\sqrt{p^+}}\right]+O\left(|\mu|^4\,e^{|t|(\frac{(b-c)}{\sqrt{p^+}}+\frac{(c-a)}{\sqrt{p^-}})}\right) \quad (31)$$

**(iv)** If $\alpha'_{21}=0$ and $\alpha'_{11}=0$, then

$$w(\mu)=\Delta_{24}\alpha'_{10}\alpha'_{20}\mu^4\cos\left[\frac{\mu\,(c-a)}{\sqrt{p^-}}\right]\cos\left[\frac{\mu\,(b-c)}{\sqrt{p^+}}\right]+O\left(\left|\mu^3\right|e^{|t|(\frac{(b-c)}{\sqrt{p^+}}+\frac{(c-a)}{\sqrt{p^-}})}\right) \quad (32)$$

Now we are ready to derived the needed asymptotic formulas for eigenvalues and eigenfunctions.

**Theorem 5.** *The boundary-value-transmission problem* (1)-(4) *has an precisely numerable many real eigenvalues, whose behavior may be expressed by two sequence* $\{\mu_{n,1}\}$ *and* $\{\mu_{n,2}\}$ *with following asymptotic as* $n \to \infty$

***(i)*** *If* $\alpha'_{21} \neq 0$ *and* $\alpha'_{11} \neq 0$*, then*

$$\mu_{n,1} = \frac{\sqrt{p^-}(n-3)\pi}{(c-a)} + O\left(\frac{1}{n}\right), \ \mu_{n,2} = \frac{\sqrt{p^+}n\pi}{(b-c)} + O\left(\frac{1}{n}\right), \tag{33}$$

***(ii)*** *If* $\alpha'_{21} \neq 0$ *and* $\alpha'_{11} = 0$*, then*

$$\mu_{n,1} = \frac{\sqrt{p^-}(2n+1)\pi}{2(c-a)} + O\left(\frac{1}{n}\right), \ \mu_{n,2} = \frac{\sqrt{p^+}(n-2)\pi}{(b-c)} + O\left(\frac{1}{n}\right), \tag{34}$$

***(iii)*** *If* $\alpha'_{21} = 0$ *and* $\alpha'_{11} \neq 0$*, then*

$$\mu_{n,1} = \frac{\sqrt{p^-}(n-2)\pi}{(c-a)} + O\left(\frac{1}{n}\right), \ \mu_{n,2} = \frac{\sqrt{p^+}(2n+1)\pi}{2(b-c)} + O\left(\frac{1}{n}\right), \tag{35}$$

***(iv)*** *If* $\alpha'_{21} = 0$ *and* $\alpha'_{11} = 0$*, then*

$$\mu_{n,1} = \frac{\sqrt{p^-}(2n-3)\pi}{2(c-a)} + O\left(\frac{1}{n}\right), \ \mu_{n,2} = \frac{\sqrt{p^+}(2n+1)\pi}{2(b-c)} + O\left(\frac{1}{n}\right) \tag{36}$$

PROOF.

Using this asymptotic expression of eigenvalues we can easily obtain the corresponding asymptotic expressions for eigenfunctions of the problem (1)-(4). Recalling that $\varphi_{\mu_{n,i}}(x)$ is an eigenfunction according to the eigenvalue $\mu_n$, and by putting (33) in the (15)-(16) for $k = 0, 1$ and denoting the corresponding eigenfunction as

$$\widetilde{\varphi}_{n,i} = \{ \begin{array}{l} \varphi^-_{\mu_{n,i}}(x) \text{ for } x \in [a,c) \\ \varphi^+_{\mu_{n,i}}(x,) \text{ for } x \in (c,b] \end{array}$$

we get the following cases If $\alpha'_{21} \neq 0$ and $\alpha_{11} \neq 0$

$$\widetilde{\varphi}_{n,1}(x) = \begin{cases} -\alpha'_{11}p^- \left[\frac{(n-3)\pi}{(c-a)}\right]^2 \cos\left[\frac{(n-3)\pi(x-a)}{(c-a)}\right] + O\left(n\right), \quad \text{for } x \in [a,c) \\ \frac{\Delta_{24}\alpha'_{11}p^-}{\Delta_{12}} \left[\frac{(n-3)\pi}{(c-a)}\right]^3 \sin\left[(n-3)\pi\right] \cos\left[\frac{\sqrt{p^-}(n-3)\pi(x-c)}{\sqrt{p^+}(c-a)}\right] \\ +O\left(n^2\right), \text{ for } x \in (c,b] \end{cases}$$

and

$$\widetilde{\varphi}_{n,2}(x)=\begin{cases} -\alpha'_{11}p^{+}\left[\frac{n\pi}{(b-c)}\right]^{2}\cos\left[\frac{\sqrt{p^{+}}n\pi(x-a)}{\sqrt{p^{-}}(b-c)}\right]+O\left(n\right), \quad \text{for } x\in[a,c) \\ \frac{\Delta_{24}\alpha'_{11}}{\Delta_{12}\sqrt{p^{-}}}\left[\frac{\sqrt{p^{+}}n\pi}{(b-c)}\right]^{3}\sin\left[\frac{\sqrt{p^{+}}n\pi(c-a)}{\sqrt{p^{-}}(b-c)}\right]\cos\left[\frac{n\pi(x-c)}{(b-c)}\right] \\ +O\left(n^{2}\right), \quad \text{for } x\in(c,b] \end{cases}$$

If $\alpha'_{21}\neq 0$ and $\alpha_{11}=0$, then

$$\widetilde{\varphi}_{n,1}(x)=\begin{cases} -\alpha'_{10}p^{-}\left[\frac{(2n+1)\pi}{2(c-a)}\right]\sin\left[\frac{(2n+1)\pi(x-a)}{2(c-a)}\right]+O\left(1\right), \quad \text{for } x\in[a,c) \\ -\frac{\Delta_{24}\alpha'_{10}p^{-}}{\Delta_{12}}\left[\frac{(2n+1)\pi}{2(c-a)}\right]^{2}\cos\left[\frac{(2n+1)\pi}{2}\right]\cos\left[\frac{\sqrt{p^{-}}(2n+1)\pi(x-c)}{2\sqrt{p^{+}}(c-a)}\right] \\ +O\left(n\right), \ \text{for } x\in(c,b] \end{cases}$$

and

$$\widetilde{\varphi}_{n,2}(x)=\begin{cases} -\alpha'_{10}\sqrt{p^{-}}\left[\frac{\sqrt{p^{+}}(n-2)\pi}{(b-c)}\right]\sin\left[\frac{\sqrt{p^{+}}(n-2)\pi(x-a)}{\sqrt{p^{-}}(b-c)}\right]+O\left(1\right), \quad \text{for } x\in[a,c) \\ -\frac{\Delta_{24}\alpha'_{10}p^{+}}{\Delta_{12}}\left[\frac{(n-2)\pi}{(b-c)}\right]^{2}\cos\left[\frac{\sqrt{p^{+}}(n-2)\pi(c-a)}{\sqrt{p^{-}}(b-c)}\right]\cos\left[\frac{(n-2)\pi(x-c)}{(b-c)}\right] \\ +O\left(n\right), \ \text{for } x\in(c,b] \end{cases}$$

If $\alpha'_{21}=0$ and $\alpha_{11}\neq 0$, then

$$\widetilde{\varphi}_{n,1}(x)=\begin{cases} -\alpha'_{11}p^{-}\left[\frac{(n-2)\pi}{(c-a)}\right]^{2}\cos\left[\frac{(n-2)\pi(x-a)}{(c-a)}\right]+O\left(n\right), \quad \text{for } x\in[a,c) \\ \frac{-\Delta_{24}\alpha'_{11}p^{-}}{\Delta_{12}}\left[\frac{(n-2)\pi}{(c-a)}\right]^{3}\sin\left[(n-2)\pi\right]\cos\left[\frac{\sqrt{p^{-}}(n-2)\pi(x-c)}{\sqrt{p^{+}}(c-a)}\right] \\ +O\left(n^{2}\right), \ \text{for } x\in(c,b] \end{cases}$$

and

$$\widetilde{\varphi}_{n,2}(x)=\begin{cases} -\alpha'_{11}p^{+}\left[\frac{(2n+1)\pi}{2(b-c)}\right]^{2}\cos\left[\frac{\sqrt{p^{+}}(2n+1)\pi(x-a)}{2\sqrt{p^{-}}(b-c)}\right]+O\left(n\right), \quad \text{for } x\in[a,c) \\ -\frac{\Delta_{24}\alpha'_{11}}{\Delta_{12}\sqrt{p^{-}}}\left[\frac{\sqrt{p^{+}}(2n+1)\pi}{2(b-c)}\right]^{3}\sin\left[\frac{\sqrt{p^{+}}(2n+1)\pi(c-a)}{2\sqrt{p^{-}}(b-c)}\right]\cos\left[\frac{(2n+1)\pi(x-c)}{2(b-c)}\right] \\ +O\left(n^{2}\right), \ \text{for } x\in(c,b] \end{cases}$$

If $\alpha'_{21}=0$ and $\alpha_{11}=0$, then

$$\widetilde{\varphi}_{n,1}(x)=\begin{cases} -\alpha'_{10}p^{-}\left[\frac{(2n-3)\pi}{2(c-a)}\right]\sin\left[\frac{(2n-3)\pi(x-a)}{2(c-a)}\right]+O\left(1\right) \quad \text{for } x\in[a,c) \\ -\frac{\Delta_{24}p^{-}\alpha'_{10}}{\Delta_{12}}\left[\frac{(2n-3)\pi}{2(c-a)}\right]^{2}\cos\left[\frac{(2n-3)\pi}{2}\right]\cos\left[\frac{\sqrt{p^{-}}(2n-3)\pi(x-c)}{2\sqrt{p^{+}}(c-a)}\right] \\ +O\left(1\right), \quad \text{for } \ x\in(c,b] \end{cases}$$

and

$$\widetilde{\varphi}_{n,2}(x)=\begin{cases} -\alpha_{10}'\sqrt{p^-}\left[\frac{\sqrt{p^+}(2n+1)\pi}{2(b-c)}\right]\sin\left[\frac{\sqrt{p^+}(2n+1)\pi(x-a)}{2\sqrt{p^-}(b-c)}\right]+O\left(1\right), & \text{for } x\in[a,c) \\ -\frac{\Delta_{24}p^+\alpha_{10}'}{\Delta_{12}}\left[\frac{(2n+1)\pi}{2(b-c)}\right]^2\cos\left[\frac{\sqrt{p^+}(2n+1)\pi(c-a)}{2\sqrt{p^-}(b-c)}\right]\cos\left[\frac{(2n+1)\pi(x-c)}{2(b-c)}\right] & \\ +O\left(\frac{1}{n}\right), \text{ for } x\in(c,b] & \end{cases}$$

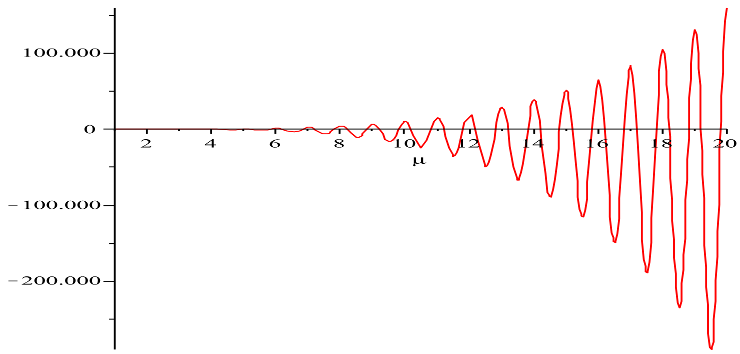
All this asymptotic approximations are hold uniformly for $x$. **Example.** Consider the following simple case of the BVTP's $(1)-(3)$

$$-y''(x)=\mu^2 y(x) \qquad x\in[-\pi,0)\cup(0,\pi] \tag{37}$$

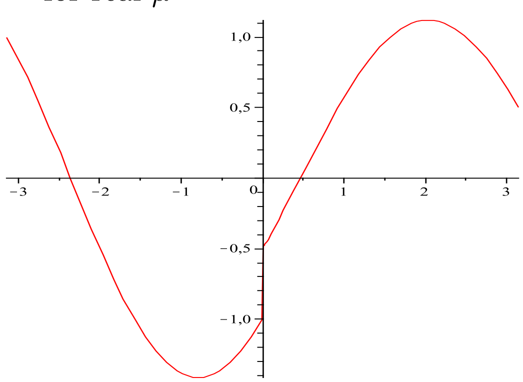
$$y(-\pi)+\mu^2 y'(-\pi)=0, \tag{38}$$

$$\mu^2 y(\pi)+y'(\pi)=0, \tag{39}$$

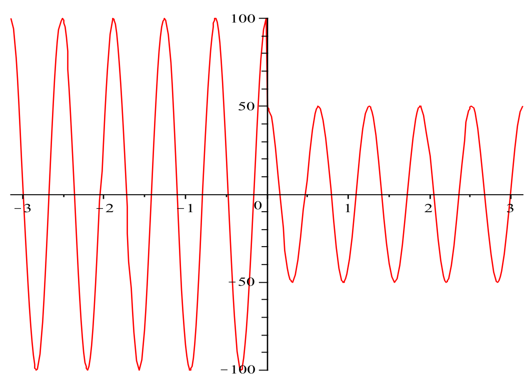
$$y(0-)=2y(+0), \quad y'(-0)=y'(+0) \tag{40}$$




`Figure`1: Graph of the function $w(\mu)$ for real $\mu$




`Figure` 2:Graph of the solution $\phi(x,\mu)$ for $\mu=1$




`Figure` 3:Graph of the solution $\phi(x,\mu)$ for $\mu=10$

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